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Note

On Cycles in Multipartite Tournaments

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An n-partite tournament is an orientation of a complete n-partite graph. We show that if D is a strongly connected n-partite tournament, and if v is the only vertex in one of the partite sets of D, then for any m, $3 \leqslant m \leqslant n$, there is an m-cycle of D containing v. This generalizes a theorem of Moon.

1. A digraph obtained by replacing each edge of a complete n-partite ($n \geqslant 2$) graph with an arc having the same terminal vertices is called an n-partite (multipartite) tournament.

Moon [1] and Bondy [2] were the first to consider cycles in the entire class of multipartite tournaments. Bondy proved (Theorem 3 of [2]) that any strongly connected n-partite ($n \geqslant 3$) tournament contains k-cycles for all $k \in \{3, 4, ..., n\}$. He showed also (Theorem 2) that if a strongly connected n-partite ($n \geqslant 5$) tournament has in each partite set at least two vertices, then it has a k-cycle with $k > n$. In connection with the last statement he asked [2] if the inequality $k > n$ may be replaced by the equality $k = n + 1$. A negative answer to this question was obtained by the author in [3] (for details see [4]). The same counterexample (as in [3, 4]) was found independently by Balakrishnan and Paulraja [5]. In [4] it was also proved that the inequality $k > n$ may be replaced by the inequality $n + 1 \leqslant k \leqslant n + 2$.

It is easy to see that Theorem 3 of [2] is a generalization of a theorem on tournaments due to Moser [6] (that states that any strongly connected tournament is pancyclic). In connection with this Bondy [2] raised the question if the corresponding genealization of Moon's theorem [7] is also true. Moon's theorem asserts that a strongly connected tournament D is

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vertex pancyclic, i.e., for each $k \in \{3, 4, ..., n\}$ and each $v \in V(D)$ the tournament D contains a k-cycle passing through v. Bondy further gave an example showing that the last generalization is not true in general.

In [8] it has been shown that every vertex of a strongly connected n-partite tournament ($n \geqslant 3$) is in a cycle that contains vertices from exactly m partite sets, for all $3 \leqslant m \leqslant n$. In this note we establish another restricted generalization of Moon's theorem.

2. We first introduce some notation. $V(D)$, $A(D)$ are the sets of vertices and arcs of a digraph D. For $W \subset V(D)$, $D\langle W \rangle$ denotes the subgraph of D induced on W. By a cycle (path) we mean a directed cycle (directed path).

For a cycle $C = (x_1, x_2, ..., x_n, x_1)$, we denote the path from x_i to x_j along C by $C[x_i, x_j]$. For a path $P = (y_1, y_2, ..., y_m)$ and $i < j$, we denote the path from y_i to y_j along P by $P[y_i, y_j]$. For X, $Y \subset V(D)$, $A(X, Y) = \{(x, y) \in A(D) : x \in X, y \in Y\}$.

For a cycle $C = (x_1, x_2, ..., x_n, x_1)$, when there is need, we assume that subscripts are taken modulo n.

3. THEOREM. *Let D be a strongly connected n-partite* ($n \geqslant 3$) *tournament, one of the partite sets of which consists of a single vertex, say, v. Then for each* $m \in \{3, 4, ..., n\}$ *in D there is an m-cycle of D containing v.*

Proof. Let $V_1 = \{v\}$, $V_2, ..., V_n$ be the partite sets of D. We prove the theorem by induction on m. As D is strongly connected, $A(\{w : (v, w) \in A(D)\}, \{u : (u, v) \in A(D)\}) \neq \varnothing$, and, therefore, D has a three-cycle running through v. Thus, for $m = 3$ the theorem is true. We next prove the theorem for $m = j$ ($4 \leqslant j \leqslant n$), assuming that it holds for $m = j - 1$. Let $C = (v_1, v_2, ..., v_{j-1}, v_1)$ be a cycle of length $j - 1$ containing v. Without loss of generality assume that $v = v_1$. As $j - 1 < n$, no vertex of V_s is on C, for some s, $2 \leqslant s \leqslant n$. Let u_1 be a vertex of V_s. Clearly, (u_1, v_i) or $(v_i, u_1) \in A(D)$, $1 \leqslant i \leqslant j - 1$. If $H = D\langle V(C) \cup \{u_1\} \rangle$ is strongly connected then there exists an $i \in \{1, 2, ..., j - 1\}$ such that (v_i, u_1), $(u_1, v_{i+1}) \in A(D)$. In this case $(u_1, C[v_{i+1}, v_i], u_1)$ is a j-cycle of D containing $v (= v_1)$.

Suppose that H is not strongly connected. Without loss of generality assume that $A(u_1, V(C)) = \varnothing$ (otherwise consider the converse digraph of D). Let $P = (u_1, u_2, ..., u_t)$, $t \geqslant 3$, be a shortest path from u_1 to C, and, hence, $u_t = v_k$ for some k, $1 \leqslant k \leqslant j - 1$.

Case 1. $u_t \neq v_1$. If $t \geqslant k$, then $(v_1, Q, C[v_{k+1}, v_1])$ is a cycle of length j containing v_1, where Q is the subpath of P having the last $k - 1$ edges of P. If $k > t$ then $(C[v_1, v_{k-t+1}], P, C[v_{k+1}, v_1])$ is a j-cycle, as required.

Case 2. $v_1 = u_t$. By the choice of P, (v_1, u_i) is in $A(D)$ for each $1 \leqslant i \leqslant t-2$. If $t \geqslant j$, then $(P[u_{t-j+1}, u_t], u_{t-j+1})$ is a cycle of length j containing v_1. If $t < j$ then $(u_1, u_2, ..., u_t, v_2, v_3, ..., v_{j-t+1}, u_1)$ is a j-cycle containing v_1. This completes the proof of the theorem. ▌

This theorem immediately implies the aforementioned result of Moon [7].

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